MODEL OF A POROUS MATERIAL CONSIDERING THE PLASTIC ZONE NEAR THE PORE

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1. Formulation of the Problem. The problem of the strain of a porous material containing a large number of spherical pores is almost impossible to solve exactly. Therefore an approximate approach is widely used in which averaging transforms the porous material to a continuous one with effective moduli of elasticity and yield surface. Today there are a large number of averaging methods [1-6].

Here the simplest method of virial decomposition is chosen, which is valid for a small volumetric pore composition, and which is accurate to terms $O(m_1^2)$, where $m_1 << 1$. The limitation comes from ignoring elastic interaction between pores in the virial decomposition [3]. The average stresses σ_{ij} and strains ε_{ij} of the porous material are determined by formulas [3, chapter 5, paragraph 4], which in our notation have the form

$$\varepsilon_{ij} = m_1\varepsilon_{ij}^0 + m_2\varepsilon_{ij}^s, \quad \sigma_{ij} = m_2\sigma_{ij}^s, \tag{1.1}$$

where ε_{ij}^0 and ε_{ij}^s are the average strains in the pore and the material; σ_{ij}^s is the average stress in the material; m_1 and m_2 are the volumetric concentrations of the pores and the material, for which the following formulas are valid

$$m_1 = \frac{4}{3}\pi a^3 n, \quad m_1 + m_2 = 1$$

where n and a are the concentration and radius of the spherical pores. In the elastic case, the magnitude of ε_{ij}^s is determined by Hooke's law [3]

$$\begin{aligned} \varepsilon_{ij}^s &= \frac{1}{3}\varepsilon_{kk}^s\delta_{ij} + e_{ij}^s, \quad e_{ij}^s = S_{ij}/(2\mu_s m_2), \\ \varepsilon_{kk}^s &= -p/(K_s m_2), \quad \sigma_{ij} = -p\delta_{ij} + S_{ij}. \end{aligned} \tag{1.2}$$

Here μ_s is the shear modulus and K_s is the bulk compression modulus of the material; p is the pressure; S_{ij} and e_{ij} are the stress and strain deviators; and δ_{ij} is the Kronecker delta. In the plastic case, ε_{ij}^s is found from the Prandtl–Reiss relationships (see paragraph 3).

The pore strain ε_{ij}^0 is determined from Eshelby's solution [2]. If the stresses are high enough, then stress concentrations around the pore create a plastic zone and the strain becomes elastic-plastic. In this case there is no exact solution, and an approximate solution must be constructed for the average ε_{ij}^0. We choose a coordinate system which coincides with the major axes of the stress and strain(tensors the material is isotropic), and represent the pore strain ε_i^0 in the form

$$\varepsilon_i^0 = \varepsilon_i^s + u_i/a, \tag{1.3}$$

where ε_i^s is the strain in the material if there were no pores, and u_i^0 is the additional displacement of the pore along the i-th axis, which displacement is related to the stress concentration. The value of ε_i^s is determined from Eqs. (1.2), and u_i^0 is deter-

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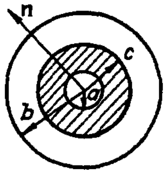

Fig. 1

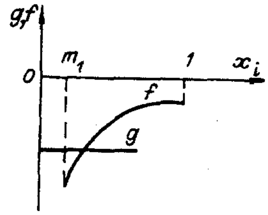

Fig. 2

mined from the approximate solution. In order to construct the approximate solution, the whole volume is divided into spherical cells such that the center of each cell has a pore of radius a [1, 7-9] (Fig. 1). The cell radius is found from the formula

$$b = a/m_1^{1/3}.$$

If the boundary conditions are constant, it is assumed [1] that a stress $\sigma_{ij}{}^{\gamma}$ that coincides with the average stress σ_{ij} is applied to the cell surface (the Reiss method [1-3]). Thus, the problem

$$\begin{aligned}
&\nabla_j\sigma'_{ij} = 0, \quad f_i\big|_{r=a} = 0, \quad f_i\big|_{r=b} = \sigma_{ij}n_j, \quad i, j = 1, 2, 3, \\
&\sigma'_{ij} = -p'\delta_{ij} + S'_{ij}, \quad \varepsilon'_{ij} = \frac{1}{3}\varepsilon'_{kk}\delta_{ij} + e'_{ij}, \quad \varepsilon'_{kk} = -p'/K_s, \\
&e'_{ij} = S'_{ij}/2\mu_s, \quad I'_2 < Y_s^2, \quad I'_2 = \frac{3}{2}S'_{ij}S'_{ij}, \\
&\dot{e}'_{ij} = \frac{1}{2\mu_s}\dot{S}'_{ij} + \dot{\lambda}S'_{ij}, \quad I'_2 = Y_s^2, \quad \varepsilon'_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)
\end{aligned} \tag{1.4}$$

must be solved in the cell $a < r < b$. Here the n_j are components of the vector normal to the cell boundary (Fig. 1), and ε_{ij}' and σ_{ij}' are the microstrains and microstresses in the cell. In the elastic-plastic case there is no exact solution to Eqs. (1.4); therefore we seek an approximate solution which satisfies the equilibrium equations only along r, and the system of equations has the form [10]

$$\frac{d\sigma'_r}{dr} + 2\frac{(\sigma'_r - \sigma'_\varphi)}{r} = 0, \quad \sigma'_\theta = \sigma'_\varphi, \quad f_r\big|_{r=a} = 0, \quad f_r\big|_{r=b} = \sigma_n, \tag{1.5}$$

where the subscripts r, θ, and φ are the components of the spherical coordinate system in the cell, and σ_n is the normal stress applied at the cell boundary:

$$\sigma_n = -p + S_1 n_1^2 + S_2 n_2^2 + S_3 n_3^2. \tag{1.6}$$

The equilibrium equations are not solved for θ and φ, and we approximately set $u_\theta = u_\varphi = 0$. The system (1.5) and (1.6) is solved in the approximation that the material is not work-hardened in the plastic region; therefore the yield strength Y_s = const. The solution to Eqs. (1.5) and (1.6) for the spherical tensor of average stresses $\sigma_n = -p$ is known [10] and is determined by the following formulas:

in the elastic case ($c < r < b$)

$$\begin{aligned}
&\sigma'_r = -p + p\left(1 - \frac{b^3}{r^3}\right)\Big/\left(1 - \frac{b^3}{a^3}\right), \quad \sigma'_\theta = \sigma'_\varphi, \\
&\sigma'_\varphi = -p + p\left(1 + \frac{b^3}{2r^3}\right)\Big/\left(1 - \frac{b^3}{a^3}\right);
\end{aligned} \tag{1.7}$$

and in the plastic case ($a < r < c$)

$$\sigma'_r = 2\varkappa Y_s \ln\frac{r}{a}, \quad \sigma'_\varphi = \varkappa Y_s + 2\varkappa Y_s \ln\frac{r}{a}, \quad \sigma'_\theta = \sigma'_\varphi, \tag{1.8}$$

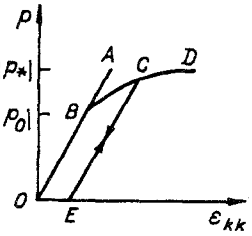

Fig. 3

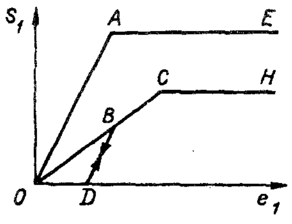

Fig. 4

where $\varkappa = 1$ in tension, $\varkappa = -1$ in compression, and c is the radius of the plastic zone (see Fig. 1). We neglect the compressibility of the material and represent the displacements in the form

$$u_r = \begin{cases} -\dfrac{p}{4\mu_s}\dfrac{b^3a^3}{(b^3-a^3)r^2}, & |p| \leq |p_0|, \\ \varkappa\dfrac{Y_s}{6\mu_s}\dfrac{c^3}{r^2}, & |p_0| < |p| < |p_*|, \end{cases}$$

$$p + \frac{2}{3}\varkappa Y_s\left(1 - \left(\frac{c}{b}\right)^3 + 3\ln\left(\frac{c}{a}\right)\right) = 0, \tag{1.9}$$

$$p_0 = -\frac{2}{3}\varkappa Y_s\left(1 - \frac{a^3}{b^3}\right), \quad p_* = -2\varkappa Y_s \ln\frac{b}{a},$$

$$\varkappa = 1, \quad p < 0; \quad \varkappa = -1, \quad p > 0.$$

The system of Eqs. (1.7)-(1.9) is valid when the inequality $|p| < |p_*|$ is satisfied. When it is not, the equilibrium equations (1.5) have no solutions. The approximate solution for a non-spherical stress tensor ($\sigma_{ij} = -p\delta_{ij} + S_{ij}$) is determined from Eqs. (1.7)-(1.9), in which we must make the substitution

$$p \to p - (S_1n_1^2 + S_2n_2^2 + S_3n_3^2).$$

As a result, the displacement of the pore surface along the i-th axis in the elastic case is given by the formula

$$u_i^0 = -\frac{p}{4\mu_s}\frac{b^3a}{(b^3-a^3)} + \frac{S_i}{4\mu_s}\frac{b^3a}{(b^3-a^3)}, \tag{1.10}$$

and in the elastic-plastic case by the formulas

$$u_i^0 = \varkappa\frac{Y_s}{6\mu_s}\frac{c_i^3}{a^2}, \quad a \leq c_i < b,$$

$$\sigma_i = \frac{2}{3}\varkappa Y_s\left(1 - \left(\frac{c_i}{b}\right)^3 + \ln\left(\frac{c_i}{b}\right)^3 + \ln\left(\frac{b}{a}\right)^3\right), \tag{1.11}$$

$$\varkappa = \begin{cases} 1, & \sigma_i > 0, \\ -1, & \sigma_i < 0. \end{cases}$$

By substituting (1.2), (1.10), and (1.11) in Eq. (1.3), we find the pore strain ε_j^0 and find the average strain ε_j from Eq. (1.1). For the elastic strain, the approximate solution (1.3) and (1.10) coincides well enough with Eshelby's exact solution [see Eqs. (2.2) and (2.3)]. There is no exact solution when a plastic zone arises. Therefore, in order to verify the method, we examine the tension in a plane with holes in Sec 2. The approximate solution (2.12) constructed by this method was compared with D. D. Ivlev's solution (2.10) [10]. The results (2.13) show that the relative error in ε_i does not exceed 16%, which is completely satisfactory.

2. **Calculation of μ and K**. Let the pressure p satisfy the inequality $|p| < |p_0|$; then the plastic zone does not arise and the cell deforms elasticly. By substituting u_i^0 from Eq. (1.10) into Eqs. (1.1) and (1.3) we use Eq. (1.2) and obtain

$$\varepsilon_i = -\frac{p}{3K_{el}} + \frac{S_i}{2\mu_{el}}, \quad K_{el} = K_s m_2 \Big/ \left(1 + \frac{m_1}{2}\frac{(1+\nu)}{(1-2\nu)}\right),$$
$$\mu_{el} = \mu_s m_2/(1 + m_1/2). \tag{2.1}$$

By expanding the expressions for [the elastic] K_{el} and μ_{el} in powers of m_1, we find

$$K_{el} = K_s\left(1 - \frac{3}{2}m_1\frac{(1-\nu)}{(1-2\nu)}\right), \quad \mu_{el} = \mu_s\left(1 - \frac{3}{2}m_1\right). \tag{2.2}$$

The corresponding elastic constants obtained with Eshelby's exact solution [2, 3] have the form

$$K_T = K_s\left(1 - \frac{3}{2}m_1\frac{(1-\nu)}{(1-2\nu)}\right), \quad \mu_T = \mu_s\left(1 - \frac{5}{3}m_1\right), \tag{2.3}$$

where the assumed $\nu = 1/2$ in calculating μ_T. Comparison of Eqs. (2.2) and (2.3) shows that the expressions for K_{el} and K_T coincide, and the difference in μ is small, on the order of $(\mu_T - \mu_{el})/\mu_{el} \approx 0.16m_1$.

If $|p| > |p_0|$, a plastic zone arises in the cell and the strains ε_i become elastic-plastic. By substituting u_i^0 from (1.11) into Eqs. (1.1) and (1.3), we obtain

$$\varepsilon_i = \frac{\sigma_i}{2\mu_s m_2} + \frac{3\nu p}{E_s m_2} + \frac{\varkappa Y_s}{6\mu_s}x_i,$$
$$\frac{3}{2}\frac{\sigma_i}{\varkappa Y_s} + \ln m_1 - 1 = \ln x_i - x_i, \quad x_i = \left(\frac{c_i}{b}\right)^3, \quad m_1 \leqslant x_i < 1. \tag{2.4}$$

The solution to the second equation, which determines x_i, can be found graphically [Fig. 2, where $g = \frac{3}{2}\sigma_i/(\varkappa Y_s) + \ln m_1 - 1$ and $f = \ln x_i - x_i$). A monotonic dependence of σ_i on x_i follows from Fig. 2:

$$\frac{d}{dx_i}\left(\frac{3}{2}\frac{\sigma_i}{\varkappa Y_s}\right) = \frac{1}{x_i} - 1 > 0, \quad m_1 \leqslant x_i < 1,$$
$$\frac{d\sigma_i}{dx_i} = 0 \quad \text{for } x_i = 1.$$

By setting $\sigma_i = -p + S_i$ in (2.4), where $|S_i/p| < 1$, we expand the right side of the first Eq. (2.4) in a power series in S_i/p; then by going from the major axes to an arbitrary axis, we obtain [7, 8]

$$e_{ij} = \frac{S_{ij}}{2\mu_p}, \quad \varepsilon_{kk} = -\frac{p}{K_p}, \quad \varepsilon_{ij} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} + e_{ij},$$
$$K_p = K_s m_2 \Big/ \left(1 - \frac{\varkappa}{3}\frac{(1+\nu)}{(1-2\nu)}\frac{Y_s}{p}m_p m_2\right), \tag{2.5}$$
$$\mu_p = \mu_s m_e \Big/ \left(\eta + \frac{1}{2}m_p\right), \quad \eta = \frac{m_e}{m_2}, \quad m_e + m_p = 1, \quad m_p = \left(\frac{c}{b}\right)^3.$$

The value of $(c/b)^3$ is determined from the second Eq. (1.9), which has the approximate solution

$$m_e = \sqrt{\frac{1 + m_1^2}{m_2} - \frac{2m_1}{m_2}\operatorname{ch}\left(\frac{3p}{2Y_s}\right)}, \quad m_e = 1 - \left(\frac{c}{b}\right)^3. \tag{2.6}$$

The strain ε_{ij} in (2.5) is the sum of the elastic strains ε_{ij}^e and the plastic strains $\varepsilon_{ij}^{p\prime}$, the latter is found from the equations

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^{p\prime}_{ij}, \quad e^e_{ij} = S_{ij}/2\mu_{el}, \quad \varepsilon^e_{kk} = -p/K_{el}. \tag{2.7}$$

Equations (2.5) are valid for the loading stage, which is determined by the condition that the work of the stresses on the plastic strains must be positive [11]:

$$\sigma_{ij}\dot{\varepsilon}^{p\prime}_{ij} > 0, \quad \dot{\varepsilon}^{p\prime}_{ij} = d\varepsilon^{p\prime}_{ij}/dt. \tag{2.8}$$

By using (2.1) and (2.5), we represent the plastic strains as

$$\varepsilon^{p\prime}_{kk} = -p\left(\frac{1}{K_p} - \frac{1}{K_{el}}\right), \quad e^{p\prime}_{ij} = \frac{S_{ij}}{2}\left(\frac{1}{\mu_p} - \frac{1}{\mu_{el}}\right).$$

By differentiating the left and right sides of these expressions with respect to t and considering the explicit forms of K_p, K_{el}, μ_p and μ_{el}, we obtain

$$\dot{e}^{p\prime}_{ij} = \frac{\dot{S}_{ij}}{4\mu_s}\frac{(m_p - m_1)}{m_2 m_e} + \frac{S_{ij}\dot{m}_p}{4\mu_s m_e^2},$$

$$\dot{\varepsilon}^{p\prime}_{kk} = \frac{1}{K_s}\left(\frac{1+\nu}{1-2\nu}\right)\left(\frac{\dot{p}m_1}{2m_2} + \frac{1}{3}\varkappa Y_s \dot{m}_p\right),$$

where $\dot{m}_p = -\frac{3}{2}\frac{\dot{p}}{\varkappa Y_s}\frac{m_p}{m_e}$ is determined from (1.9) and (2.5). By rewriting the inequality (2.8) in the form

$$-p\dot{\varepsilon}^{p\prime}_{kk} + S_{ij}\dot{e}^{p\prime}_{ij} > 0$$

and substituting the expressions for $\dot{e}_{ij}^{p\prime}$ and $\dot{\varepsilon}_{kk}^{p\prime}$ into it, we find the load condition and the condition for the applicability of the system (2.5):

$$\begin{gathered} \dot{I} > 0, \quad p_0^2 < I_1 < p_*^2, \quad \dot{I} = \frac{dI}{dt}, \\ \dot{I} = \left(\frac{3}{2}\zeta + m_p I_2/(2m_e^3 Y_s\sqrt{I_1})\right)\dot{I}_1 + \zeta\dot{I}_2/3, \\ \zeta = \frac{m_p}{m_e} - \frac{m_1}{m_2}, \quad I_1 = p^2, \quad I_2 = \frac{3}{2}S_{ij}S_{ij}. \end{gathered} \tag{2.9}$$

In the other cases the strain is elastic and is described by Hooke's law with constants μ_{el} and K_{el} [see (2.1)]. The fact that the inequality (2.8) is satisfied shows that this model satisfies the second law of thermodynamics, because the entropy change is determined by the formula [12]

$$\dot{S} = \frac{1}{\rho T}\sigma_{ij}\dot{\varepsilon}^{p\prime}_{ij}.$$

In order to estimate the error of this method, we examine the problem of stretching a plain with circular holes of radius a, where the hole region is completely plastic. The material strain ε_i^T in this case is determined by D. D. Ivlev's formulas (see (8.27) from [10]); in our notation it has the form

$$\begin{gathered} e_i^T = \frac{S_i}{2\mu_s}\left(1 + \frac{2c^2}{r^2} - \frac{c^4}{r^4}\right), \quad \varepsilon^T_{kk} = \frac{3u_0}{r}, \quad u_0 = \frac{kc^2}{2\mu_s r}, \\ k = Y_s/\sqrt{3} \quad (S_i/k < 1). \end{gathered} \tag{2.10}$$

We find the approximate value of ε_i^{appr} by using a method given in [10]. Under a pressure p, the material displacement is [10]

$$u = u_0, \quad c^2 = a^2/\exp(1 + p/k), \tag{2.11}$$

where u_0 is determined in (2.10). We replace $p \to p - S_i$ in (2.11) and expand [the result] in a Taylor series in powers of (S_i/k), and consider (1.1) and (1.3) to obtain

$$e_i^{\mathrm{appr}} = \frac{S_i}{2\mu_s}\left(1 + \frac{c^2}{r^2}\right), \quad \varepsilon_{kk}^{\mathrm{appr}} = \frac{3u_0}{r}. \tag{2.12}$$

From Eqs. (2.10) and (2.12) we find the relative error in determining the strain:

$$\frac{\delta e_i}{e_i^{\mathrm{appr}}} = x\left(\frac{1-x}{1+x}\right), \quad x = \frac{c^2}{r^2}, \quad \delta e_i = e_i^{\mathrm{appr}} - e_i^{\mathrm{T}}, \quad m_1 = \frac{a^2}{r^2}. \tag{2.13}$$

From this it follows that the maximum relative error does not exceed 0.16.

3. **Plasticity**. Up to now, we have considered the case where the average stresses do not lie on the yield surface and the occurrence of plastic deformation is related to the microstress concentrations near the pore. After the average stresses S_{ij} reach the yield surface, plastic strains $\varepsilon_{ij}^{p''}$ arise, which are determined by the associative flow yield [7-9]

$$\dot{\varepsilon}_{ij}^{p''} = \dot{\lambda}' \frac{\partial \Phi}{\partial \sigma_{ij}}, \tag{3.1}$$

where Φ is the yield surface, which is found from the equation $\Phi(I_1, I_2, m_1) = 0$. The total strain ε_{ij} is represented as a sum

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p'} + \varepsilon_{ij}^{p''}, \tag{3.2}$$

where ε_{ij}^{e} and $\varepsilon_{ij}^{p'}$ are found from (2.5) and (2.7). By following [9], we write Φ as a function of I_1, I_2, and m_1 in the form $\Phi = \frac{3}{2} S_{ij}S_{ij} - Y^2(p, m_1)$. In order to determine Y^2, we make use of the fact that the microstresses lie on the Mises yield surface $\frac{3}{2} S'_{ij}S'_{ij} = Y_s^2$ during plastic yield. By averaging this formula over the material volume, we obtain

$$\frac{3}{2} S_{ij}S_{ij} + \frac{3}{2} m_2^2 \langle S''_{ij}S''_{ij}\rangle_s = m_2^2 Y_s^2,$$
$$S_{ij} = m_2 \langle S'_{ij}\rangle_s, \quad S''_{ij} = S'_{ij} - \langle S'_{ij}\rangle_s, \quad \langle S''_{ij}\rangle_s = 0. \tag{3.3}$$

The quantity $\langle S''_{ij}S''_{ij}\rangle_s$ is the average of the solution (1.7) and (1.8) in the cell:

$$\langle S''_{ij}S''_{ij}\rangle_s = \begin{cases} \frac{3}{2} p^2 \frac{m_1}{m_2}, & |p| < |p_0|, \\ \frac{2}{3} Y_s^2 \left(1 - \frac{m_e^2}{m_2}\right), & |p_0| < |p| < |p_*|. \end{cases}$$

By substituting $\langle S''_{ij}S''_{ij}\rangle_s$ into (3.3), we find the yield surface of the porous material:

$$\Phi = \frac{3}{2} S_{ij}S_{ij} - Y^2 = 0,$$
$$Y^2 = \begin{cases} Y_s^2 m_2^2 - \frac{9}{4} p^2 m_1, & |p| \leqslant |p_0|, \\ Y_s^2 m_2 m_e^2, & |p_0| < |p| \leqslant |p_*|. \end{cases} \tag{3.4}$$

Direct numerical calculations have shown [7, 8] that (3.4) coincides with Garson's formula [9]. As Tvergaard noted [13], m_1 must be replaced by k^*m_1 in Garson's formula for cylindrical pores and also in (3.4), where $k^* = 1.5$. For special pores it is recommended that $k^* = 1.7$. The factor k^* is related to the pressure of regions in the cell which are not considered in deriving (3.3). By differentiating (3.2) with respect to time and considering (2.5), (2.7), (3.1), and (3.4), we obtain a Prandtl–Reiss type equation

$$\frac{1}{2\mu}\overset{\nabla}{S}_{ij} + \dot{\lambda} S_{ij} = \dot{e}_{ij}, \quad \dot{\varepsilon}_{kk} = -\left(\frac{p_x}{K}\right)^{\cdot} + \frac{1}{3}\dot{\lambda}\frac{\partial Y^2}{\partial p},$$
$$\frac{1}{2\mu}\overset{\nabla}{S}_{ij} = \left(\frac{S_{ij}}{2\mu}\right)^{\cdot} - \frac{1}{2\mu}(\omega_{ik}S_{kj} + \omega_{jk}S_{ik}), \quad \omega_{ij} = \frac{1}{2}(\nabla_i v_j - \nabla_j v_i). \tag{3.5}$$

Here $\mu = \mu_p$ and $K = K_p$ for $\dot{I} > 0$ and $p_0^2 < I_1 < p_*^2$; while $\mu = \mu_{el}$ and $K = K_{el}$ in other cases; $\dot{\lambda}$ is determined from the condition

$$d\Phi = d\left(\frac{3}{2}S_{ij}S_{ij} - Y^2(p, m_1, Y_s)\right) = 0,$$

which leads to the equation

$$3S_{ij}\frac{dS_{ij}}{d\lambda} - \frac{\partial Y^2}{\partial p}\frac{dp}{d\lambda} + \frac{\partial Y^2}{\partial m_1}\frac{\partial m_2}{\partial \lambda} - \frac{\partial Y^2}{\partial Y_s}\frac{dY_s}{d\lambda} = 0. \tag{3.6}$$

By multiplying the first Eq. (3.5) by S_{ij} and considering that $\Phi = 0$, we obtain

$$S_{ij}\frac{dS_{ij}}{d\lambda} = 2\mu S_{ij}\frac{de_{ij}}{d\lambda} - \frac{4}{3}\mu Y^2 + \frac{2}{3}\frac{Y^2}{\mu}\frac{dp}{d\lambda}. \tag{3.7}$$

By substituting (3.7) and the second Eq. (3.5) into (3.6), we find

$$d\lambda = \left(6\mu S_{ij}de_{ij} + \left(2\frac{Y^2}{\mu}\frac{d\mu}{dp} - \frac{\partial Y^2}{\partial p}\right)dp - \right.$$
$$\left. - \frac{\partial Y^2}{\partial Y_s}dY_s\right) \Big/ \left(4\mu Y^2 + \frac{m_2}{3}\frac{\partial Y^2}{\partial m_1}\frac{\partial Y^2}{\partial p}\right). \tag{3.8}$$

Closure of (3.8) requires specifying a hardening law $Y_s = Y_s(\varepsilon^p_{ij}\varepsilon^p_{ij})$ and an equation of state $p = p(\rho, \mathscr{E})$. Here we consider an ideal plastic material, so Y_s = const. The equation of state is given in the usual form that considers thermal effects

$$p = p_x + p_T, \quad \mathscr{E} = \mathscr{E}_x + \mathscr{E}_T, \quad p_T = \Gamma\rho\mathscr{E}_T,$$
$$\Gamma/\Gamma_s = K/(m_2K_s), \quad \mathscr{E}_T = c_V T,$$
$$\mathscr{E}_x = \left(\frac{1}{2}K_{el}(\varepsilon^e_{kk})^2 + \mu_{el}e^e_{ij}e^e_{ij}\right)/\rho, \; p_x = -K_{el}\varepsilon^e_{kk},$$

where T is the temperature, c_V is the heat capacity; $\mathscr{E}$, $\mathscr{E}_x$, and $\mathscr{E}_T$ are the specific energy, the "cold" energy and the thermal energy; p_x and p_T are the cold and thermal pressure; and Γ is the Grüneisen coefficient. The increment in the specific entropy for plastic flow is determined by the equation [12] $dS = \frac{1}{\rho T}\sigma_{ij}d\varepsilon^p_{ij}$, which along with (3.2) gives

$$dS = \frac{1}{\rho T}\sigma_{ij}d\varepsilon^{p\prime}_{ij} + \frac{1}{\rho T}\sigma_{ij}d\varepsilon^{p\prime\prime}_{ij}. \tag{3.9}$$

As was shown above in Eq. (2.8), the inequality $\sigma_{ij}d\varepsilon^{p\prime}_{ij} > 0$ is valid when $\dot{I} > 0$ and $p_0^2 < I_1 < p_*^2$; in other cases $\sigma_{ij}d\varepsilon^{p\prime}_{ij} = 0$, because $d\varepsilon^{p\prime}_{ij} = 0$. Therefore the first term in (3.9) is greater than or equal to zero. We now clarify the sign of the second term. By determining $d\varepsilon^{p\prime\prime}_{ij}$ from (3.1) and considering (3.4), we find

$$\sigma_{ij}d\varepsilon^{p\prime\prime}_{ij} = d\lambda\left(\frac{2}{3}Y^2 - \frac{p}{3}\frac{\partial Y^2}{\partial p}\right), \quad d\lambda > 0. \tag{3.10}$$

The yield surface (3.4) satisfies the inequality $p(\partial Y^2/\partial p) \leq 0$; therefore Eq. (3.10) is greater than zero. If $I_2 < Y^2$ then $d\varepsilon_{ij}^{p''} = 0$ and $\sigma_{ij} d\varepsilon_{ij}^{p''} = 0$, so that $\sigma_{ij} d\varepsilon_{ij}^{p''} \geq 0$ in the general case. By substituting these expressions into (3.9) we obtain $dS \geq 0$.

4. **Discussion of Results**. Based on two examples, we examine the basic features of a porous material within the framework of the model presented here. In the first example we will assume that the stress tensor is spherical ($\sigma_{ij} = -p\delta_{ij}$). Then the function $p(\varepsilon_{kk})$ for the load with a plastic zone is determined from the first and third Eqs. (2.5). This dependence is shown as the curve OBC in Fig. 3. Unloading from the state C occurs along the line $\Delta p = -K_{el}\Delta\varepsilon_{kk}$. Subsequent loading to the point C also occurs along the line EC (self-loading [11]). At the point D ($p = |p_*|$) we have $dp/d\varepsilon_{kk} = 0$, while for $p > |p_*|$ the pore becomes unstable and the pore is compressed (or grows). If the plastic zone is not considered, then loading and unloading occur along the line OA, which is described by the equation $p = -K_{el}\varepsilon_{kk}$.

In the second example we first apply a constant pressure p to the porous material, and then apply shear stresses S_i : $S_3 = 0$ and $S_1 + S_2 = 0$. Two cases of loading from the same porosity m_1 are represented in the S_1, e_1 plane in Fig. 4. In the first case, $p = p_1 = \text{const}$ and $|p_1| < |p_0|$, as shown by the lines OAE, so the plastic zone does not arise. When S_1 is shifted to the point A, loading and unloading are described by the equation $S_1 = 2\mu_{el}e_1$. Plastic yield starts at point A, therefore $S_1 = \frac{1}{\sqrt{3}} Y(p_1, m_1) = \text{const}$ on the line AE. In the second case $|p_2| > |p_0|$, $p_2 = \text{const}$, a plastic zone arises, and the loading on the segment OC satisfies the equation $S_1 = 2\mu_p e_1$. Unloading from point B is determined by the equation $\Delta S_1 = 2\mu_{el}\Delta e_1$. Subsequent loading to point B is also described by this equation. Plastic flow (3.1) with a constant stress $S_1 = \frac{1}{\sqrt{3}} Y(p_2, m_1)$, starts at the point C, where $Y(p, m_1)$ is found from (3.4). The positions of the lines OAE and OCH in Fig. 4 are determined by the inequalities $|p_2| > |p_1|$, $Y(p_2, m_1) < Y(p_1, m_1)$, and $\mu_p < \mu_{el}$. We note that if the plastic zone is not considered, then the loading curve coincides with OAE. By using Eqs. (2.1), (2.5), and (2.7), we define the plastic strain $e_1^{p'}$ on the segment OC (Fig. 4) in the form

$$e_1^{p'} = \frac{S_1}{2\mu_p} - \frac{S_1}{2\mu_{el}} = \frac{S_1}{2\mu_s}\frac{(m_p - m_1)}{m_e m_2}. \tag{4.1}$$

At point C, according to (3.4),

$$S_1 = \frac{Y_s}{\sqrt{3}}\sqrt{m_2}\, m_e. \tag{4.2}$$

By substituting (4.2) into (4.1), at point C we obtain

$$e_1^{p'}\big|_c = \frac{Y_s}{2\sqrt{3}\,\mu_s}\frac{(m_p - m_1)}{\sqrt{m_2}},$$

from which the maximum strain at $m_p = 1$ is

$$(e_1^{p'})_{\max} = \frac{Y_s\sqrt{m_2}}{2\sqrt{3}\,\mu_s}. \tag{4.3}$$

It follows from Figs. 3 and 4 and an evaluation of (4.3) that considering the plastic zone qualitatively changes the loading curve of a porous material.

The plastic zone also has a large effect on the yield strength as a function of pressure, which is determined by Eqs. (3.4). If this effect is not considered, then Y(p) has the form

$$Y^2 = Y_s^2 m_2^2 - \frac{9}{4}p^2 m_1. \tag{4.4}$$

It follows from Eq. (4.4) that $Y = 0$ for $p_1^* = \frac{2}{3} Y_s \frac{m_2}{m_1^{1/2}}$, while from (3.4) we have $Y = 0$ for $p_2^* = \frac{2}{3} Y_s \ln\frac{1}{m_1}$. As a result we obtain that the ratio of critical pressures $p_1^*/p_2^* \to \infty$ as $m_1 \to 0$.

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